CHAPTER 3

Mathematical Reasoning

CBSE Class 11 · Applied Mathematics (241) · Unit 3

CBSE · Applied Mathematics · Class 11

WHAT THIS CHAPTER DOES

A Tell a 'statement' apart from a question, command, opinion or open sentence — and give its truth value.

B Write the negation of any statement, including quantified ones ('all', 'some', 'there exists').

Boards prep that builds confidence, not anxiety.

TODAY'S MISSION

Today's mission

1. Tell a 'statement' apart from a question, command, opinion or open sentence — and give its truth value.
2. Write the negation of any statement, including quantified ones ('all', 'some', 'there exists').
3. Decide the truth value of compound statements built with AND and (inclusive) OR.
4. Write the converse, inverse and contrapositive of a conditional, and validate or disprove a statement.

WHY THIS MATTERS

Why this chapter matters

1. A compact 6-mark unit that is almost pure logic — high return for a couple of hours of clean practice.
2. The reasoning here (if-then, contrapositive, counter-example) is the skeleton of every proof you will meet in Calculus, Statistics and beyond.
3. Outside the exam: precise logical language is the working tool of law, economics, computer science and clear argument of every kind.

TOPIC

A

Statements and their negation

TOPIC

Statements, non-statements and negation

WHAT A STATEMENT IS

In mathematical reasoning a STATEMENT is a declarative sentence that is DEFINITELY TRUE or DEFINITELY FALSE — never both at once and never ambiguous. 'The sum of the angles of a triangle is 180°' is a statement (true); 'New Delhi is the capital of France' is a statement (false). The single test is:

WHAT IS NOT A STATEMENT

Questions ('What is the value of x?'), commands ('Solve the equation'), exclamations ('How beautiful!'), requests and opinions ('Mathematics is the best subject') are NOT statements, because no fixed true/false value can be attached. An OPEN SENTENCE such as '$x + 3 = 7$' is also not a

NEGATION OF A STATEMENT

The NEGATION of a statement p, written ~p (read 'not p'), is the statement whose truth value is the OPPOSITE of p's. If p is true, ~p is false, and if p is false, ~p is true. We usually form it by inserting 'not' or the phrase 'It is not the case that ...'. Example: p = 'New York is a city' (true);

TRUTH VALUE, STATED CLEANLY

Every statement carries a TRUTH VALUE: T (true) or F (false). In an exam, when asked 'is this a statement, and what is its truth value?', answer in two clean parts: first whether it qualifies as a statement (declarative, unambiguous, fixed truth value), then T or F with a one-line reason. For a

WORKED EXAMPLE

Classify each sentence and negate the statements

1. QUESTION: For each, say whether it is a statement; if so give its truth value and its negation. (a) '7 is a prime number.' (b) 'Open the window.' (c) '$x + 5 = 9$.' (d) 'The number 10 is divisible by 4.'
2. (a) STATEMENT, truth value TRUE (7 has no factors other than 1 and itself). Negation: '7 is NOT a prime number' (false).
3. (b) NOT a statement — it is a command, so no truth value can be assigned.
4. (c) NOT a statement — it is an open sentence; its truth depends on x (true for $x = 4$, false otherwise). It becomes a statement only once x is fixed.
5. (d) STATEMENT, truth value FALSE ($10 \div 4 = 2.5$, not a whole number). Negation: '10 is NOT divisible by 4' (true).
6.

TOPIC

B

Compound statements — AND and OR

THEOREM · LOAD-BEARING RESULT

Logical connectives — conjunction (AND) and disjunction (OR)

"A COMPOUND statement is formed by joining two (or more) statements with a connective. CONJUNCTION 'p AND q' ($p \wedge q$) is TRUE only when BOTH p and q are true; otherwise it is false. DISJUNCTION 'p OR q' ($p \vee q$), in the INCLUSIVE sense used in mathematics, is TRUE when p is true, or q is true, or both; it is false only when BOTH are false.

STATEMENT

Truth tables. AND: (T,T)→T, (T,F)→F, (F,T)→F, (F,F)→F. OR (inclusive): (T,T)→T, (T,F)→T, (F,T)→T, (F,F)→F. To analyse a compound statement, first identify the COMPONENT statements and the

WHY THIS MATTERS

- Almost all precise reasoning combines simple statements: 'a number is divisible by 2 AND by 3', 'x is a vowel OR x is the letter y'
- Knowing exactly when a conjunction or disjunction is true lets you assign a single truth value to a complicated sentence — and lets you split a compound

WATCH OUT FOR

NOTE Two opposite traps: (i) AND needs BOTH parts true — one false component makes the whole thing false; (ii) OR is INCLUSIVE — 'both true' still makes OR true, and OR is false ONLY when both parts fail. Do not import everyday English's occasional exclusive 'or' (as in 'tea or coffee') into mathematics.

WORKED EXAMPLE

Form compound statements and find their truth values

1. QUESTION: Let p: '2 is an even number' (T) and q: '3 is an even number' (F). Find the truth value of (a) $p \wedge q$, (b) $p \vee q$. Also (c) split 'The number 6 is divisible by 2 and by 3' into components and connective.

2. (a) $p \wedge q$ ('2 is even AND 3 is even'): AND is true only if BOTH are true. Here q is false, so $p \wedge q$ is FALSE.

3. (b) $p \vee q$ ('2 is even OR 3 is even'): inclusive OR is true if at least one part is true. Here p is true, so $p \vee q$ is TRUE.

4. (c) Components: r = '6 is divisible by 2' (T) and s = '6 is divisible by 3' (T); connective = AND. Since both are true, $r \wedge s$ is TRUE.

5. TAKEAWAY: AND demands both parts true; inclusive OR demands at least one. Identify the components and the connective first, then read the truth value straight off the rule.

TOPIC

C

Quantifiers — 'there exists' and 'for all'

THEOREM · LOAD-BEARING RESULT

The quantifiers ∀ and ∃ — and how to negate them

> A QUANTIFIER states 'how many'. The UNIVERSAL quantifier 'for all / for every' (symbol ∀) asserts a property holds for EVERY element. The EXISTENTIAL quantifier 'there exists / for some' (symbol ∃) asserts a property holds for AT LEAST ONE element. Their negations swap type: the negation of 'for all x, P(x)' is 'there exists x such that NOT P(x)'; the negation of 'there exists x, P(x)' is 'for all x,

STATEMENT

$\sim(\forall x, P(x)) \equiv \exists x, \sim P(x)$ and $\sim(\exists x, P(x)) \equiv \forall x, \sim P(x)$. In words: the negation of 'ALL A are B' is 'SOME A are NOT B' (equivalently 'at least one A is not B'); the negation of 'SOME A are B' is 'NO A is B' (= 'ALL

WHY THIS MATTERS

- Quantifiers make everyday words like 'every', 'some', 'at least one' precise, which is exactly what mathematical statements need
- Negating quantified statements correctly is one of the most-tested ideas in this unit, because it is so easy to get wrong by reflex.

WATCH OUT FOR

NOTE The classic error: the negation of 'ALL students passed' is NOT 'all students failed' — it is 'SOME student did not pass' (one exception is enough). Negation flips ∀ to ∃-with-not and ∃ to ∀-with-not; it does NOT simply flip the verb while keeping the quantifier.

WORKED EXAMPLE

Negate quantified statements correctly

1. QUESTION: Write the negation of (a) 'Every student in the class has a smartphone.' (b) 'There exists a real number whose square is negative.'

2. (a) This is UNIVERSAL ('every'). Negation = existential-of-the-opposite: 'There exists at least one student in the class who does NOT have a smartphone' (equivalently 'Some student in the class does not have a smartphone').

3. (b) This is EXISTENTIAL ('there exists'). Negation = universal-of-the-opposite: 'For every real number, its square is NOT negative' (equivalently 'No real number has a negative square'). [This negation is in fact TRUE, since $x^2 \geqslant 0$ for all real x.]

4. CHECK the rule: $\sim(\forall P) = \exists(\sim P)$ and $\sim(\exists P) = \forall(\sim P)$. Both answers follow it exactly.

5. TAKEAWAY: to negate a quantified statement, SWAP the quantifier ($\forall \leftrightarrow \exists$) and negate the predicate — never keep the same quantifier and merely flip the verb.

TOPIC

D

Implications — if-then, only if, if-and-only-if

THEOREM · LOAD-BEARING RESULT

Conditional, biconditional and the necessary/sufficient language

> A CONDITIONAL (implication) 'if p then q', written $p \Rightarrow q$, claims that whenever p holds, q holds. Equivalent phrasings: 'p implies q', 'p only if q', 'q if p', 'p is sufficient for q', 'q is necessary for p'. A BICONDITIONAL 'p if and only if q', written $p \Leftrightarrow q$, means BOTH $p \Rightarrow q$ and $q \Rightarrow p$ hold — i.e. q is NECESSARY AND SUFFICIENT for p.

STATEMENT

$p \Rightarrow q$ is false in exactly one case: p true and q false (a true hypothesis cannot give a false conclusion); it is true in all other rows. Phrase dictionary: 'p only if q' $\equiv p \Rightarrow q$ (q is NECESSARY for p); 'p if

WHY THIS MATTERS

- Implications are the engine of mathematics — every theorem is an 'if ..
- then ...'
- Reading 'only if', 'if', and 'if and only if' precisely (and knowing which way the arrow points) is worth the largest single block of marks in this unit

WATCH OUT FOR

NOTE 'p ONLY IF q' means $p \Rightarrow q$ (NOT $q \Rightarrow p$) — the word 'only' makes q the necessary condition. 'p IF q' means $q \Rightarrow p$. Mixing these reverses the implication. Also, 'if and only if' is a DOUBLE implication — proving only one direction is a half-answer.

THEOREM · LOAD-BEARING RESULT

Converse, inverse and contrapositive of $p \Rightarrow q$

> From a conditional $p \Rightarrow q$ we build three related conditionals: the CONVERSE $q \Rightarrow p$ (swap the parts); the INVERSE $\sim p \Rightarrow \sim q$ (negate both parts); and the CONTRAPOSITIVE $\sim q \Rightarrow \sim p$ (negate AND swap). The contrapositive is LOGICALLY EQUIVALENT to the original — they always share the same truth value. The converse and the inverse are NOT equivalent to the original (though the converse

STATEMENT

Original: $p \Rightarrow q$. Converse: $q \Rightarrow p$. Inverse: $\sim p \Rightarrow \sim q$. Contrapositive: $\sim q \Rightarrow \sim p$. Equivalences: $(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p)$ [original ≡ contrapositive]; $(q \Rightarrow p) \equiv (\sim p \Rightarrow \sim q)$ [converse ≡ inverse]. Worked instance —

WHY THIS MATTERS

- Writing these three forms is the single most-asked 3-mark question in the chapter
- The contrapositive's equivalence is also the basis of an entire proof technique (proof by contrapositive): to prove $p \Rightarrow q$, you may instead prove $\sim q \Rightarrow \sim p$.

WATCH OUT FOR

NOTE Do NOT confuse the converse (swap only) with the contrapositive (negate AND swap). Memory hook: Converse = Change the order; coNtrapositive = Negate aNd swap; Inverse = Invert (negate) both, same order. Only the contrapositive matches the original's truth value.

WORKED EXAMPLE

Write all three forms and translate 'only if'

1 QUESTION (a): For 'If it is raining, then the ground is wet', write the converse, inverse and contrapositive.

2 (a) Let p: 'it is raining', q: 'the ground is wet' (statement is $p \Rightarrow q$). CONVERSE ($q \Rightarrow p$): 'If the ground is wet, then it is raining.' INVERSE ($\sim p \Rightarrow \sim q$): 'If it is not raining, then the ground is not wet.' CONTRAPOSITIVE ($\sim q \Rightarrow \sim p$): 'If the ground is not wet, then it is not raining.'

3 (a) Note: the contrapositive is logically equivalent to the original; the converse is not (the ground could be wet for another reason).

4 QUESTION (b): Rewrite using 'if ... then ...': 'A triangle is equilateral only if all its angles are equal.'

5 (b) 'p only if q' means $p \Rightarrow q$. So: 'IF a triangle is equilateral, THEN all its angles are equal.' (Here 'all angles equal' is the NECESSARY condition for being equilateral.)

6

TOPIC

E

Validating statements

TOPIC

How to decide whether a statement is true

DIRECT METHOD

To validate a conditional 'if p then q' by the DIRECT method, you ASSUME that p is true and then, by valid steps, SHOW that q must follow. This is the workhorse of ordinary proof: start from the hypothesis, use known facts and algebra, and arrive at the conclusion. Example: to show 'if n is

METHOD OF CONTRAPOSITIVE

Because a conditional $p \Rightarrow q$ is LOGICALLY EQUIVALENT to its contrapositive $\sim q \Rightarrow \sim p$, you may prove the statement by instead assuming $\sim q$ (the conclusion is false) and showing $\sim p$ (the hypothesis must be false). This is useful when assuming 'not q'

METHOD OF CONTRADICTION

In proof by CONTRADICTION you ASSUME the statement is FALSE and then derive an impossibility (a contradiction), which forces the original statement to be true. The classic Class 10 example is $\sqrt{2}$ being irrational: assume it is rational, written p/q in lowest

DISPROOF BY COUNTER-EXAMPLE

To DISPROVE a UNIVERSAL statement ('for all x, P(x)') you need just ONE counter-example: a single x for which P(x) is false. One exception is enough — you do NOT have to overturn every case. Example: 'every prime number is odd' is disproved by the single

WORKED EXAMPLE

Validate one statement, disprove another

1. QUESTION (a): Validate 'If n is an odd integer, then n^2 is odd' by the direct method.

2. (a) Assume n is odd, so $n = 2k + 1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is of the form $2m + 1$ — an odd number. Hence the statement is TRUE.

3. QUESTION (b): Check whether 'For every real number x, $x^2 > x$' is true; if not, disprove it.

4. (b) This is a UNIVERSAL statement, so one counter-example disproves it. Take $x = 1/2$: $x^2 = 1/4 = 0.25$ and $x = 0.5$, so $x^2 = 0.25$ is NOT greater than 0.5. The statement FAILS for $x = 1/2$, hence it is FALSE.

5. TAKEAWAY: to confirm a universal, give a general argument (direct/contrapositive/contradiction); to deny a universal, a single concrete counter-example is enough and is the cleaner route.

TRY IT · SOLVE BEFORE YOU PEEK

(a) Negate: 'All birds can fly.' (b) Write the contrapositive of 'If x is divisible by 4, then x is divisible by 2.' (c) Is 'p only if q' the same as $p \Rightarrow q$ or $q \Rightarrow p$?

SOLUTION

ANSWER (a) 'Some bird cannot fly' (equivalently 'there exists a bird that cannot fly') — NOT 'all birds cannot fly'. (b) Contrapositive: 'If x is NOT divisible by 2, then x is NOT divisible by 4' ($\sim q \Rightarrow \sim p$). (c) 'p only if q' means $p \Rightarrow q$ (q is the necessary condition).

TOPIC

What counts as a statement

TRAP → TRUTH

× **MISTAKE** Any sentence in mathematics is a 'statement', including questions like 'What is the value of x?' or commands like 'Solve the equation'.

✓ **CORRECT** A STATEMENT is a declarative sentence that is DEFINITELY TRUE or DEFINITELY FALSE — never both, and never ambiguous. Questions ('What is x?'), commands ('Solve this'), exclamations and opinions ('Mathematics is beautiful') are NOT statements. Also, an open sentence like '$x + 3 = 7$' is NOT a statement until x is fixed, because its truth depends on the value of x. The test: can you, in principle, assign exactly one truth value (T or F)? If yes, it is a statement.

TOPIC

Negation of a quantified statement (ALL / SOME)

TRAP → TRUTH

× **MISTAKE** The negation of 'All students passed' is 'All students failed' (just flip the verb).

✓ **CORRECT** The negation of 'ALL students passed' is 'SOME students did NOT pass' (equivalently, 'There is at least one student who did not pass'). The negation of a UNIVERSAL ('for all') is an EXISTENTIAL of the opposite ('there exists … not'), and vice versa: the negation of 'SOME students passed' is 'NO student passed' = 'ALL students did not pass'. Flipping 'passed' to 'failed' while keeping 'all' is the classic error — 'all passed' being false does NOT require everyone to fail, just one person.

TOPIC

Inclusive vs exclusive OR

TRAP → TRUTH

× **MISTAKE** 'p OR q' means exactly one of p, q is true (one or the other but not both).

✓ **CORRECT** In mathematics 'OR' is INCLUSIVE: 'p OR q' ($p \vee q$) is TRUE when p is true, OR q is true, OR BOTH are true. It is false only when BOTH are false. Example: 'A number is divisible by 2 OR by 3' is true for 6 (divisible by both). Everyday English sometimes uses 'or' exclusively ('tea or coffee'), but the mathematical default is inclusive.

TOPIC

Truth value of AND (conjunction)

TRAP → TRUTH

× **MISTAKE** 'p AND q' is true if at least one of p, q is true.

✓ **CORRECT** 'p AND q' ($p \wedge q$) is TRUE only when BOTH p and q are true; it is FALSE if even one of them is false. That is the opposite end from OR. Example: 'Delhi is in India AND 2 + 2 = 5' is FALSE, because the second part is false — one false component makes the whole conjunction false.

TOPIC

Converse vs contrapositive

TRAP → TRUTH

× **MISTAKE** The contrapositive of 'if p then q' is 'if q then p' (just swap the two parts).

✓ **CORRECT** SWAPPING gives the CONVERSE ($q \Rightarrow p$). The CONTRAPOSITIVE negates AND swaps: '$\sim q \Rightarrow \sim p$'. The contrapositive is LOGICALLY EQUIVALENT to the original (same truth value always); the converse is NOT. Memory hook for the four forms of $p \Rightarrow q$: Converse = swap ($q \Rightarrow p$); Inverse = negate both ($\sim p \Rightarrow \sim q$); Contrapositive = negate AND swap ($\sim q \Rightarrow \sim p$). Only the contrapositive matches the original's truth value.

TOPIC

Meaning of 'p only if q'

TRAP → TRUTH

× **MISTAKE** 'p only if q' means the same as 'if q then p' ($q \Rightarrow p$).

✓ **CORRECT** 'p ONLY IF q' translates to 'if p then q' ($p \Rightarrow q$) — q is a NECESSARY condition for p. By contrast 'p IF q' means 'if q then p' ($q \Rightarrow p$) — q is SUFFICIENT for p. And 'p IF AND ONLY IF q' combines both: $p \Leftrightarrow q$ (q is necessary AND sufficient). Reading 'only if' as ordinary 'if' reverses the implication and loses the mark.

TOPIC

Disproving a universal statement

TRAP → TRUTH

× **MISTAKE** To disprove 'every prime number is odd' you must show that NO prime is odd.

✓ **CORRECT** To DISPROVE a UNIVERSAL statement ('for all x, P(x)') you need only ONE counter-example — a single x for which P(x) fails. 'Every prime is odd' is disproved by the single prime 2 (which is even). You do NOT have to overturn every case; one exception is enough. Conversely, to PROVE a universal you must cover ALL cases, but to prove an EXISTENTIAL ('there exists') a single example suffices.

TOPPER TEMPLATE · MARK-BY-MARK

3 marks: 'Write the converse, inverse and contrapositive of the statement: "If a number is

1	**IDENTIFY P AND Q CLEARLY** 1 m	Let p: 'a number is divisible by 6' and q: 'the number is divisible by 3'. The given statement is the conditional p ⇒ q ('if p then q'). Naming p and q first prevents swapping errors in the next step.
2	**WRITE THE CONVERSE AND THE INVERSE** 1 m	CONVERSE (swap, q ⇒ p): 'If a number is divisible by 3, then it is divisible by 6.' INVERSE (negate both, ~p ⇒ ~q): 'If a number is NOT divisible by 6, then it is NOT divisible by 3.'
3	**WRITE THE CONTRAPOSITIVE AND NOTE THE EQUIVALENCE** 1 m	CONTRAPOSITIVE (negate and swap, ~q ⇒ ~p): 'If a number is NOT divisible by 3, then it is NOT divisible by 6.' Remark: the contrapositive is logically equivalent to the original (both true here); the converse is NOT (e.g. 9 is divisible by 3 but not by 6, so the converse is false).

TOPPER TEMPLATE · MARK-BY-MARK

2 marks: 'Write the negation of each statement: (a) "All teachers are punctual." (b) "Some

1. **NEGATE THE UNIVERSAL STATEMENT** — 1 m

   (a) The statement is universal ('all'). Its negation is existential-of-the-opposite: 'There exists at least one teacher who is NOT punctual', equivalently 'Some teachers are not punctual.' (NOT 'all teachers are not punctual' — one exception is enough to make 'all are punctual' false.)

2. **NEGATE THE EXISTENTIAL STATEMENT** — 1 m

   (b) The statement is existential ('some'). Its negation is universal-of-the-opposite: 'NO student likes mathematics', equivalently 'All students do NOT like mathematics.' Rule: negation flips ∀ to ∃-not and ∃ to ∀-not.

TOPPER TEMPLATE · MARK-BY-MARK

3 marks: 'Check whether the statement is true: "For every real number x, x^2 is greater than x." If

1 **CLASSIFY THE STATEMENT AND CHOOSE A METHOD** 1 m

The statement is UNIVERSAL: 'for every real number x, $x^2 > x$'. To DISPROVE a universal statement it is enough to produce ONE counter-example — a single x for which $x^2 > x$ fails. (To prove it we would need all x; to disprove, one suffices.)

2 **PRODUCE A COUNTER-EXAMPLE** 1 m

Take $x = 1/2$. Then $x^2 = 1/4 = 0.25$, and $x = 0.5$. Here $x^2 = 0.25$ is NOT greater than $x = 0.5$ — in fact $0.25 < 0.5$. So the statement fails for $x = 1/2$. ($x = 1$ also works: $1^2 = 1$ is not greater than 1.)

3 **STATE THE CONCLUSION** 1 m

Since there exists a real number ($x = 1/2$) for which $x^2 > x$ is false, the universal statement 'for every real x, $x^2 > x$' is FALSE. One counter-example is sufficient to disprove it.

PYQ PATTERNS

Top PYQ patterns to drill

#1	Write the converse, inverse and contrapositive of a given conditional 'if p then q'. (3 marks)	Almost every annual paper + SQP
#2	Write the negation of a given statement, including statements that contain a quantifier ('all', 'some', 'there exists'). (1-2 marks)	SQP + annual pattern
#3	Identify whether a given sentence is a statement, and give its truth value with a reason. (1 marks)	Annual
#4	Form the compound statement using AND / OR and decide its truth value, or break a compound statement into its components and the connective. (2 marks)	Annual
#5	Check the validity of a statement (direct, contrapositive, or contradiction) or disprove a universal statement by a counter-example. (3 marks)	Most years

RECAP · MEMORISE THESE

Recap — the logic toolkit

1 Statements — A statement is definitely true or definitely false. Questions, commands, opinions and open sentences are NOT statements. Negation ~p has the opposite truth value.

2 AND / OR — p ∧ q is true only if BOTH are true. p ∨ q (inclusive) is true unless BOTH are false. Identify components + connective, then read the truth value.

3 Quantifiers — ∀ = for all; ∃ = there exists. ~(∀ P) = ∃(~P) ('not all' = 'some not'); ~(∃ P) = ∀(~P) ('none' = 'all not').

4 Implications — 'if p then q' = p ⇒ q; 'p

5 Converse / inverse / contrapositive —

WHAT'S NEXT

What's next

→ Unit 4 — Calculus (functions, limits and the idea of the derivative).

→ Sit the 15-MCQ Quick Drill for this chapter — under 18 minutes, target ⩾ 12/15.

→ Then the full school-pattern paper — 30 marks, internal-exam style, with full model answers.

You've mastered Unit 3 — Mathematical Reasoning.

Statements · negation · AND/OR · quantifiers · if-then · converse, inverse & contrapositive — now prove it on the paper.