CHAPTER 3

Calculus — Differentiation, Applications & Integration

CBSE Class 12 Applied Mathematics · Unit 3

CBSE · Applied Mathematics · Class 12

WHAT THIS CHAPTER DOES

A Differentiate any cost or revenue function and read $MC = dC/dx$ and $MR = dR/dx$ fluently.

B Find the profit-maximising output using $P'(x) = 0$ confirmed by the second-derivative test $P''(x) < 0$.

Boards prep that builds confidence, not anxiety.

TODAY'S MISSION

Today's mission

1. Differentiate any cost or revenue function and read $MC = dC/dx$ and $MR = dR/dx$ fluently.
2. Find the profit-maximising output using $P'(x) = 0$ confirmed by the second-derivative test $P''(x) < 0$.
3. Evaluate a definite integral by substitution — remembering to change the limits.
4. Compute consumer surplus and producer surplus from a demand/supply function and the market price.

WHY THIS MATTERS

Why this chapter matters

1. Worth ~15 marks — the single heaviest unit in Class 12 Applied Maths. Maxima-minima alone is a guaranteed long-answer every year.
2. Calculus is the mathematics of CHANGE: differentiation measures rates (marginal cost, slope), integration accumulates totals (area, surplus).
3. Every later economics application — optimisation, surplus, growth models — is built on the differentiation and integration learned here.

TOPIC

A

Differentiation & higher-order derivatives

THEOREM · LOAD-BEARING RESULT

The derivative as a rate of change

> " The derivative f'(x) = dy/dx of a function y = f(x) measures the instantaneous rate of change of y with respect to x — geometrically, the slope of the tangent to the curve at that point.

STATEMENT

For y = f(x), the derivative is f'(x) = dy/dx. Power rule: $d/dx(x^n) = n \cdot x^{n-1}$. Sum rule: the derivative of a sum is the sum of the derivatives. The SECOND derivative f''(x) = d/dx[f'(x)] measures the rate of

WHY THIS MATTERS

- Every commerce application of calculus reads a derivative as a RATE
- Marginal cost is the rate of change of total cost
- marginal revenue the rate of change of revenue

WATCH OUT FOR

NOTE Apply the power rule precisely: $d/dx(x^n) = n \cdot x^{n-1}$ — the exponent drops by ONE and multiplies in front. A constant term differentiates to ZERO. The commonest slip is mishandling fractional or negative coefficients on the cubic and quadratic terms of cost functions.

WORKED EXAMPLE

Worked example — second derivative & concavity

1. QUESTION: For $f(x) = x^3 - 3x^2 + 4$, find $f'(x)$ and $f''(x)$, and state where the curve is concave up.

2. STEP 1 — First derivative: $f'(x) = 3x^2 - 6x$ (power rule term by term; the constant 4 differentiates to 0).

3. STEP 2 — Second derivative: $f''(x) = 6x - 6$ (differentiate $f'(x)$: $d/dx(3x^2) = 6x$, $d/dx(-6x) = -6$).

4. STEP 3 — Concavity: the curve is concave UP where $f''(x) > 0$, i.e. $6x - 6 > 0 \Rightarrow x > 1$. It is concave down for $x < 1$, with a point of inflexion at $x = 1$ (where $f'' = 0$ and changes sign).

5. TAKEAWAY: the second derivative's SIGN encodes the shape — positive means a valley-like (concave up) region, negative means a hill-like (concave down) region. This is exactly the information the maxima-minima test needs.

TOPIC

B

Applications — maxima & minima

THEOREM · LOAD-BEARING RESULT

The second-derivative test for optimisation

> To find the local maxima and minima of $f(x)$: first solve $f'(x) = 0$ to locate the critical points; then at each critical point $x = c$, evaluate $f''(c)$. If $f''(c) < 0$ the point is a LOCAL MAXIMUM; if $f''(c) > 0$ it is a LOCAL MINIMUM; if $f''(c) = 0$ the test is inconclusive.

STATEMENT

Step 1: $f'(x) = 0 \Rightarrow$ critical points. Step 2 (classify): $f''(c) < 0 \Rightarrow$ local MAX (concave down, hill-top); $f''(c) > 0 \Rightarrow$ local MIN (concave up, valley-bottom); $f''(c) = 0 \Rightarrow$ inconclusive (check sign of f'

WHY THIS MATTERS

- Maxima-minima is the heaviest sub-topic in this unit (~26% of marks) and appears as a long-answer EVERY year — usually disguised as 'maximise profit' or 'minimise cost'
- The second-derivative test is the standard, mark-earning method examiners expect to see.

WATCH OUT FOR

NOTE $f'(x) = 0$ alone does NOT prove a maximum. You MUST classify with f''. The single biggest optimisation error is assuming every critical point is a maximum and skipping the second-derivative test — examiners reserve a full mark for that step.

WORKED EXAMPLE

Worked example — minimising average cost

1. QUESTION: The total cost of producing x units is $C(x) = x^2 + 64x + 100$ (in ₹). Find the output x that minimises the AVERAGE cost per unit, and the minimum average cost.
2. STEP 1 — Average cost: $A(x) = C(x)/x = (x^2 + 64x + 100)/x = x + 64 + 100/x = x + 64 + 100 \cdot x^{-1}$.
3. STEP 2 — Differentiate: $A'(x) = 1 - 100/x^2$ (since $d/dx(100 \cdot x^{-1}) = -100 \cdot x^{-2} = -100/x^2$; the constant 64 gives 0).
4. STEP 3 — Set $A'(x) = 0$: $1 - 100/x^2 = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10$ (reject $x = -10$ as output must be positive).
5. STEP 4 — Second-derivative test: $A''(x) = 200/x^3$ (since $d/dx(-100 \cdot x^{-2}) = 200 \cdot x^{-3}$). At $x = 10$, $A''(10) = 200/1000 = 0.2 > 0 \Rightarrow$ MINIMUM. ✓
6. STEP 5 — Minimum average cost: $A(10) = 10 + 64 + 100/10 = 10 + 64 + 10 =$ ₹84 per unit. ANSWER: produce 10 units; minimum average cost = ₹84/unit.

TOPIC

C

Marginal cost & marginal revenue

THEOREM · LOAD-BEARING RESULT

Marginal analysis — derivatives as 'one more unit'

> Marginal cost MC = dC/dx is the approximate cost of producing ONE MORE unit; marginal revenue MR = dR/dx is the approximate revenue from selling one more unit. Profit P(x) = R(x) – C(x) is maximised where MR = MC and P''(x) < 0.

STATEMENT

MC = dC/dx; MR = dR/dx. Revenue R(x) = p·x where p is price. Profit P(x) = R(x) – C(x); P'(x) = MR – MC. Setting P'(x) = 0 gives the profit-max condition MR = MC, confirmed by P''(x) < 0. Note: marginal

WHY THIS MATTERS

- Marginal cost and marginal revenue together carry ~18% of the unit's marks and are the bridge between calculus and economics
- The profit-max condition MR = MC is one of the most-tested ideas in the whole Applied-Maths course.

WATCH OUT FOR

NOTE Do NOT confuse marginal cost MC = dC/dx with average cost C(x)/x — they are different functions. And maximise PROFIT (R – C), not revenue: the revenue-maximising output and the profit-maximising output are generally different.

TOPIC

Reading marginal quantities correctly

MARGINAL COST = DERIVATIVE OF TOTAL COST

Marginal cost is MC = dC/dx, the derivative of the total cost function; it approximates the cost of producing one additional unit once x units are already being made. For $C(x) = 0.1x^2 + 20x + 500$, differentiation gives $MC = 0.2x + 20$, so at $x = 50$ the marginal cost is $0.2(50) + 20 =$ ₹30 — the

MARGINAL REVENUE = DERIVATIVE OF TOTAL REVENUE

Marginal revenue MR = dR/dx is the derivative of total revenue R(x), where revenue is price times quantity, $R(x) = p \cdot x$. When price depends on quantity through a demand relation such as $p = 100 - 2x$, revenue becomes $R(x) = (100 - 2x) \cdot x = 100x - 2x^2$, and differentiating gives MR =

PROFIT IS MAXIMISED AT MR = MC

Profit $P(x) = R(x) - C(x)$ reaches its maximum where the derivative $P'(x) = MR - MC$ equals zero, i.e. where marginal revenue equals marginal cost, provided the second derivative $P''(x)$ is negative there. Intuitively, as long as one more unit earns more than it costs (MR > MC), producing it

MARGINAL VERSUS AVERAGE — THE RECURRING TRAP

The most common marginal-analysis error in board answers is confusing marginal cost MC = dC/dx with average cost $A(x) = C(x)/x$. They answer different questions: average cost spreads the TOTAL cost (including fixed cost) over all units, while marginal cost isolates the

TOPIC

D

Integration — indefinite & definite

THEOREM · LOAD-BEARING RESULT

Integration — reversing differentiation

> Integration reverses differentiation. The INDEFINITE integral $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$ and C is an arbitrary constant. The DEFINITE integral $\int$ from a to b of $f(x)\,dx = F(b) - F(a)$ is a single number (the constant cancels).

STATEMENT

Power rule for integration: $\int x^n\,dx = x^{n+1}/(n+1) + C$ (for $n \neq -1$). Fundamental Theorem: if $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$. Techniques: substitution (let $u = g(x)$; for DEFINITE integrals change the

WHY THIS MATTERS

- Integration carries ~22% of the unit's marks and is the gateway to applications-of-integrals (area, consumer/producer surplus)
- A definite integral computes accumulated totals — areas, total cost from marginal cost, total revenue from marginal revenue.

WATCH OUT FOR

NOTE INDEFINITE integrals MUST carry + C; omitting it loses a mark. DEFINITE integrals have limits, evaluate to a number, and take NO + C. And when substituting in a definite integral, CHANGE THE LIMITS to the new variable — plugging old x-limits into a u-expression is a classic, marks-costing error.

WORKED EXAMPLE

Worked example — definite integral by substitution

1. QUESTION: Evaluate ∫ from 0 to 1 of $2x(x^2 + 1)^3$ dx.

2. STEP 1 — Substitute: let $u = x^2 + 1$, so $du = 2x\,dx$. The integrand $2x(x^2+1)^3\,dx$ becomes $u^3\,du$.

3. STEP 2 — Change the limits (definite integral!): when $x = 0$, $u = 0^2 + 1 = 1$; when $x = 1$, $u = 1^2 + 1 = 2$. New integral: ∫ from u=1 to u=2 of u^3 du.

4. STEP 3 — Integrate: $\int u^3\,du = u^4/4$. Evaluate from 1 to 2: $[2^4/4] - [1^4/4] = 16/4 - 1/4 = 4 - 0.25 = 15/4$.

5. ANSWER: $\int_0^1 2x(x^2+1)^3\,dx = 15/4 = 3.75$. KEY POINT: because this is a definite integral, we changed the limits to $u = 1$ and $u = 2$ and wrote NO constant of integration.

TOPIC

E

Applications of integrals — surplus

THEOREM · LOAD-BEARING RESULT

Consumer surplus and producer surplus

> At market equilibrium (quantity x_0, price p_0): CONSUMER SURPLUS is the area between the demand curve and the price line, CS = ∫ from 0 to x_0 of $[D(x) - p_0]$ dx. PRODUCER SURPLUS is the area between the price line and the supply curve, PS = ∫ from 0 to x_0 of $[p_0 - S(x)]$ dx.

STATEMENT

CS = $\int_0^{x_0} [D(x) - p_0]\,dx = \int_0^{x_0} D(x)\,dx - p_0 \cdot x_0$, where D(x) is the demand (price) function. PS = $\int_0^{x_0} [p_0 - S(x)]\,dx = p_0 \cdot x_0 - \int_0^{x_0} S(x)\,dx$, where S(x) is the supply function. The equilibrium (x_0,

WHY THIS MATTERS

- Consumer/producer surplus has been a fixture since 2022 and reliably contributes a 4-mark question
- It is the cleanest 'applications of integrals' topic in the commerce stream and ties calculus directly to economic welfare.

WATCH OUT FOR

NOTE Consumer surplus is NOT the whole area under the demand curve — it is the area ABOVE the price line and below the demand curve. Always subtract the price line (p_0) before integrating, and always find the equilibrium quantity x_0 first to set the upper limit.

TOPIC

Computing surplus — a clean four-step method

STEP 1 — FIND THE EQUILIBRIUM POINT

Every surplus problem begins by locating the equilibrium quantity x_0 and price p_0. If the market price is given directly, substitute it into the demand function to find x_0; for example with $D(x) = 50 - 2x$ and $p_0 = 30$, solving $30 = 50 - 2x$ gives $x_0 = 10$. If both demand and supply

STEP 2 — WRITE THE CORRECT SURPLUS INTEGRAL

Consumer surplus uses $CS = \int_0^{x_0} [D(x) - p_0]\, dx$ — the demand price MINUS the market price, integrated from zero to the equilibrium quantity. Producer surplus is the mirror image, $PS = \int_0^{x_0} [p_0 - S(x)]\, dx$ — the market price MINUS the supply price. The single most common mistake is

STEP 3 — INTEGRATE AND EVALUATE

With the integrand set up as a simple polynomial in x, apply the power rule term by term and evaluate between 0 and x_0. For consumer surplus with integrand $(20 - 2x)$, the antiderivative is $20x - x^2$, and evaluating from 0 to 10 gives $(200 - 100) - 0 = 100$. Because the integral is DEFINITE, no

STEP 4 — INTERPRET WITH UNITS AND A SANITY CHECK

Always state the surplus in rupees and give a one-line economic interpretation: consumer surplus is the extra value buyers received over what they paid, producer surplus the bonus sellers earned over their minimum acceptable price. A quick sanity check for a LINEAR

TOPIC

Second derivative test for maxima / minima

TRAP → TRUTH

× **MISTAKE** If f'(x) = 0 at a point, that point is automatically a maximum.

✓ **CORRECT** f'(x) = 0 only locates a CRITICAL point — it could be a maximum, a minimum, or neither (a point of inflexion). To classify it, use the second-derivative test: at the critical point x = c, if f''(c) < 0 the point is a LOCAL MAXIMUM (curve is concave down, like a hill-top); if f''(c) > 0 it is a LOCAL MINIMUM (concave up, like a valley); if f''(c) = 0 the test is inconclusive and you must check the sign of f' on either side. Treating every f'(x) = 0 as a maximum is the single biggest source of lost optimisation marks.

TOPIC

Marginal cost is not average cost

TRAP → TRUTH

× **MISTAKE** Marginal cost = total cost ÷ number of units = $C(x)/x$.

✓ **CORRECT** Marginal cost is the DERIVATIVE of the total cost function: $MC = dC/dx$ — the approximate cost of producing ONE MORE unit. The quantity $C(x)/x$ is the AVERAGE cost per unit, a different idea entirely. Example: if $C(x) = 500 + 20x + 0.1x^2$, then $MC = dC/dx = 20 + 0.2x$, whereas average cost $= 500/x + 20 + 0.1x$. Confusing the two is the most common marginal-analysis error.

TOPIC

Increasing / decreasing — sign of the derivative

TRAP → TRUTH

× **MISTAKE** A function is increasing wherever the function value f(x) is positive.

✓ **CORRECT** A function is increasing where its DERIVATIVE $f'(x) > 0$ (the slope is positive, the graph rises left-to-right), and decreasing where $f'(x) < 0$ (slope negative, graph falls). The SIGN OF f(x) ITSELF is irrelevant — a function can be negative and still increasing (e.g. $f(x) = x$ for $x < 0$). Always test the sign of $f'(x)$, never the sign of $f(x)$.

TOPIC

Constant of integration in indefinite integrals

TRAP → TRUTH

× **MISTAKE** $\int f(x)\,dx$ gives a single function; no constant is needed.

✓ **CORRECT** An INDEFINITE integral $\int f(x)\,dx$ always carries an arbitrary constant + C, because differentiating any constant gives zero — infinitely many functions share the same derivative. Omitting + C costs a mark on indefinite integrals. A DEFINITE integral, however, has fixed limits and evaluates to a NUMBER, so the constant cancels and is not written. Know which case you are in.

TOPIC

Definite integral limits and substitution

TRAP → TRUTH

× **MISTAKE** When you substitute $u = g(x)$ in a definite integral, you keep the original x-limits.

✓ **CORRECT** When substituting in a DEFINITE integral, you must CHANGE THE LIMITS to the new variable (compute u at the lower and upper x-limits), OR convert back to x before substituting the limits. Plugging the old x-limits into a u-expression is a classic error. Example: for ∫ from 0 to 1 of $2x(x^2+1)^3\,dx$ with $u = x^2+1$, the limits become $u = 1$ (at $x=0$) and $u = 2$ (at $x=1$).

TOPIC

Consumer surplus — which area, which price

TRAP → TRUTH

× **MISTAKE** Consumer surplus is the total area under the demand curve.

✓ **CORRECT** Consumer surplus (CS) is the area between the DEMAND curve and the horizontal line at the EQUILIBRIUM PRICE p_0, from $x = 0$ to the equilibrium quantity x_0. Formula: CS = ∫ from 0 to x_0 of [demand(x) − p_0] dx, equivalently $\int_0^{x_0} D(x)\,dx - p_0 \cdot x_0$. It measures the extra value consumers received over what they paid — NOT the entire area under the demand curve. Producer surplus is the mirror image, between the supply curve and the price line.

TOPIC

Profit maximisation condition

TRAP → TRUTH

× **MISTAKE** Profit is maximised where revenue R(x) is largest.

✓ **CORRECT** Profit P(x) = R(x) – C(x) is maximised where P'(x) = 0, i.e. where R'(x) = C'(x), that is MARGINAL REVENUE = MARGINAL COST (MR = MC), AND P''(x) < 0. Maximum revenue and maximum profit are DIFFERENT output levels in general, because producing the revenue-maximising quantity may cost more than it earns at the margin. Always optimise PROFIT, not revenue, unless the question explicitly asks for revenue.

TOPPER TEMPLATE · MARK-BY-MARK

3 marks: 'The total cost of producing x units is $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the

1 DIFFERENTIATE THE COST FUNCTION (1 m)

Marginal cost is MC = dC/dx. Differentiate term by term: $d/dx(0.005x^3) = 0.015x^2$; $d/dx(-0.02x^2) = -0.04x$; $d/dx(30x) = 30$; $d/dx(5000) = 0$. Hence $MC = 0.015x^2 - 0.04x + 30$.

2 SUBSTITUTE X = 50 (1 m)

At x = 50: $MC = 0.015(50^2) - 0.04(50) + 30 = 0.015(2500) - 2 + 30 = 37.5 - 2 + 30 =$ ₹65.50.

3 INTERPRET IN WORDS (1 m)

Interpretation: when 50 units are already being produced, the approximate cost of producing the 51st (one more) unit is ₹65.50. Marginal cost is the per-unit cost AT THE MARGIN, not the average cost over all 50 units.

TOPPER TEMPLATE · MARK-BY-MARK

5 marks: 'A firm's revenue is $R(x) = 60x - x^2$ and cost is $C(x) = x^2 + 20x + 100$ (₹, x = units). Find maximum profit.'

1 — WRITE THE PROFIT FUNCTION (1 m)

Profit $P(x) = R(x) - C(x) = (60x - x^2) - (x^2 + 20x + 100) = 60x - x^2 - x^2 - 20x - 100 = -2x^2 + 40x - 100$.

2 — DIFFERENTIATE AND SET TO ZERO (1 m)

$P'(x) = -4x + 40$. Set $P'(x) = 0$: $-4x + 40 = 0 \Rightarrow 4x = 40 \Rightarrow x = 10$. So $x = 10$ is the only critical point.

3 — APPLY THE SECOND-DERIVATIVE TEST (1 m)

$P''(x) = -4 < 0$ for all x. Since $P''(10) = -4 < 0$, the critical point $x = 10$ is a LOCAL MAXIMUM. (Equivalently MR = MC: $R'(x) = 60 - 2x$, $C'(x) = 2x + 20$; setting equal: $60 - 2x = 2x + 20 \Rightarrow 40 = 4x \Rightarrow x = 10$. ✓)

4 — COMPUTE THE MAXIMUM PROFIT (1 m)

Maximum profit $= P(10) = -2(10^2) + 40(10) - 100 = -2(100) + 400 - 100 = -200 + 400 - 100 =$ ₹100.

5 — STATE THE CONCLUSION (1 m)

Therefore the firm should produce 10 units to maximise profit, and the maximum profit is ₹100. Check via MR = MC confirmed $x = 10$, so the answer is consistent by two independent methods.

TOPPER TEMPLATE · MARK-BY-MARK

4 marks: 'The demand function is $p = 50 - 2x$ and the market (equilibrium) price is $p_0 = 30$.

1 — FIND THE EQUILIBRIUM QUANTITY (1 m)

At the market price $p_0 = 30$, the demand curve gives the quantity sold: $30 = 50 - 2x \Rightarrow 2x = 20 \Rightarrow x_0 = 10$ units. So at equilibrium, quantity $x_0 = 10$ and price $p_0 = 30$.

2 — WRITE THE CONSUMER-SURPLUS INTEGRAL (1 m)

Consumer surplus CS = ∫ from 0 to x_0 of [demand(x) − p_0] dx = ∫ from 0 to 10 of [(50 − 2x) − 30] dx = ∫ from 0 to 10 of (20 − 2x) dx.

3 — INTEGRATE (1 m)

$\int (20 - 2x)\,dx = 20x - x^2$. Evaluate from 0 to 10: $[20(10) - 10^2] - [20(0) - 0^2] = (200 - 100) - 0 = 100$.

4 — STATE THE ANSWER WITH UNITS (1 m)

Consumer surplus = ₹100. This is the total extra value the consumers gained — the difference between what they were willing to pay (the area under the demand curve up to 10 units) and what they actually paid (price 30 × 10 units = 300).

PYQ PATTERNS

Top PYQ patterns to drill

#1	Find the marginal cost and/or marginal revenue at a given output level by differentiating C(x) or R(x), then interpret the value. (3 marks)	Annual
#2	Find the value of x that maximises profit or revenue (set MR = MC, or set derivative = 0 and apply the second-derivative test). (4-5 marks)	Annual
#3	Determine the interval(s) on which a given function is increasing or decreasing. (3 marks)	Most years
#4	Evaluate a definite integral (often by substitution) and/or use it to find an area / total quantity. (3-4 marks)	Annual
#5	Compute consumer surplus or producer surplus given a demand or supply function and the market price. (4 marks)	2022, 2023, 2024, 2025

MARKS DISTRIBUTION

10-year marks distribution

10-YEAR PYQ MARKS DISTRIBUTION

Higher-order derivatives	8%
Increasing / decreasing functions	12%
Maxima and minima (incl. economic optimisation)	26%
Marginal cost and marginal revenue	18%
Indefinite and definite integrals	22%
Applications of integrals — consumer / producer surplus	14%

RECAP · MEMORISE THESE

Recap — the calculus spine

1 Differentiation — $d/dx(x^n) = n \cdot x^{n-1}$. The derivative is a RATE (slope). Second derivative f'' gives concavity.

2 Maxima / minima — Solve $f'(x) = 0$; classify with f'': $f'' < 0 \Rightarrow$ MAX, $f'' > 0 \Rightarrow$ MIN. Then compute the value.

3 Marginal analysis — $MC = dC/dx$, $MR = dR/dx$. Profit $P = R - C$ is max where $MR = MC$ and $P'' < 0$. $MC \neq$ average cost.

4 Integration — $\int x^n\,dx = x^{n+1}/(n+1) + C$. Indefinite $\Rightarrow + C$; definite $\Rightarrow F(b) - F(a)$,

5 Surplus — $CS = \int_0^{x_0} [D(x) - p_0]dx$; $PS = \int_0^{x_0} [p_0 - S(x)]dx$. Find equilibrium x_0 first.

WHAT'S NEXT

What's next

→ Chapter 4 — Probability Distributions (random variables, expectation, binomial & Poisson).

→ Sit the 15-MCQ Quick Drill for this Calculus chapter.

→ Then the full Board-Pattern Paper — 30 marks.

You've mastered Unit 3 — the heaviest unit of the paper.

Marginal analysis · maxima-minima · integration · surplus — now prove it on the board paper.

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Boards prep that builds confidence, not anxiety.